3. ANALYTICAL DESCRIPTION OF TIME AND SPECTRAL CHARACTERISTICS OF ULTRAWIDEBAND SIGNALS

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3.1 Introduction

A theoretical analysis of UWB signals can provide important insights into how UWB emissions affect various types of RF communications devices. In addition to allowing for direct calculation of interference effects, analytical results can be used to aid in the planning, design, and validation of measurements. This section details the results obtained from an analysis of proposed UWB pulse position modulation schemes.

The approach used in the analysis and the results are presented in this section. The mathematical details will be published elsewhere.

3.2 Power Spectrum of UWB Signals

The power spectral density is the average power in the signal per unit bandwidth and hence provides important information on the distribution of power over the RF spectrum. The power spectral density for a UWB pulse position modulation scheme using short duration pulses transmitted at some nominal pulse repetition rate (PRR) is given in this section. The pulse position is randomized or *dithered* with respect to the nominal pulse period. The randomization scheme analyzed in this section is referred to as *fixed time-base dither*.

3.2.1 UWB Signals Using Fixed Time-base Dither

In the fixed time-base dither scheme, each pulse occurs at the nominal pulse period, T, minus a time increment randomly distributed over a fraction of the nominal period as given in Equation 3.1. This expression also includes binary pulse modulation as proposed for communications applications.

$$x(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{1} \alpha_{kn} p_k(t - nT - \theta_n) \quad , \tag{3.1}$$

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where p_k represents the pulse shape that corresponds to an information bit (e.g., p_0 represents the value 0, p_1 represents the value 1). The coefficients α_{kn} are related to whether the n^th information bit a_n has the value 0 or 1 as follows:

$$\alpha_{kn} = \begin{cases} 1 - a_n & k = 0 \\ a_n & k = 1 \end{cases}$$

$$a_n = \begin{cases} 0 \ \ with\ prob\ \ g_0 \\ 1 \ \ with\ prob\ \ g_1 = 1 - g_0 \end{cases} \quad , \tag{3.2}$$

where g_k are the information bit probabilities (i.e., g_0 is the probability of a bit having the value 0, and $g_1 = 1 - g_0$ is the probability of a bit having the value 1). Finally, the random variables θ_n define the pulse randomization or *dithering* and are described by a density function $q(\theta)$, where

$$\wp r\ \{\theta \le \Theta \le \theta + d\theta\} = q(\theta)\, d\theta \quad . \tag{3.3}$$

For fixed time-base dither, the random variables θ_n and a_n are each assumed to be independent and identically distributed (iid).

It should be noted that the signal given in Equation 3.1 is quite general in terms of the pulse shape, binary modulation method, and pulse randomization statistics. Hence, the results presented in this section can be used to predict the power spectral density at various points in the radio link between an interfering UWB transmitter and a victim receiver (e.g., at the output of the UWB transmitter, the UWB signal radiated from a particular antenna, or in the IF section of a *narrowband* RF receiver). When dealing with linear systems, the various pulse shapes are simply related by convolutions with the appropriate transfer functions.

The power spectral density is the Fourier transform of the autocorrelation function. The autocorrelation function is obtained by taking the expected value of the signal at two different times which is expressed mathematically as

$$\begin{aligned} r_{xx}(t,s) &= \mathscr{E}\{x(t)x(s)\} = \\ &\mathscr{E}\Big\{\sum_n \sum_m \sum_k \sum_\ell \alpha_{kn}\alpha_{\ell m}\, p_k(t - nT - \theta_n)\, p_\ell(s - mT - \theta_m)\Big\} \quad . \end{aligned} \tag{3.4}$$

Taking the expectation in Equation 3.4 yields

$$r_{xx}(t,s) = \frac{1}{T^2}\left\{\sum_n \left|\sum_{k=0}^{1} g_k P_k\left(\frac{n}{T}\right)\right|^2 \left|Q\left(\frac{n}{T}\right)\right|^2 e^{i2\pi n\tau/T} + \right.$$
$$\left.\sum_{n\neq -m}\left(\sum_{k=0}^{1} g_k P_k\left(\frac{n}{T}\right)\right)\left(\sum_{k=0}^{1} g_k P_k\left(\frac{m}{T}\right)\right) Q\left(\frac{n}{T}\right) Q\left(\frac{m}{T}\right) e^{i2\pi(nt+ms)/T}\right\}$$
$$+ \tag{3.5}$$
$$\frac{1}{T}\sum_n e^{i2\pi ns/T}\left\{Q\left(\frac{n}{T}\right)\sum_{k=0}^{1} g_k p_k(\tau)\circledast p_k(-\tau)e^{i2\pi n\tau/T} - \right.$$
$$\left.\left(\sum_{k=0}^{1} g_k p_k(\tau)\circledast\sum_{\ell=0}^{1} g_\ell p_\ell(-\tau)e^{i2\pi n\tau/T}\right)\circledast\left(q(\tau)\circledast q(-\tau)e^{i2\pi\tau/T}\right)\right\}$$

were the symbol $\circledast$ is the convolution operator and $\tau = s - t$ is the time lag. Functions given in upper case letters (P, Q) are the Fourier transforms of the pulse and dithering functions.

The statistics for this process are periodic with period T as is evidenced by Equation 3.5. Such processes are commonly referred to as *cyclostationary*. Essentially this means that the statistics depend upon when the process is observed during a period. The victim receiver may observe the process at an arbitrary time during a period and hence it is useful (and simplifying) to calculate the average over all possible observation times within a period. Taking the time average over one period and the Fourier transform of Equation 3.5 yields the average power spectral density of the fixed time-base dithered UWB signal

$$\bar{R}_{xx}(f) = L + C$$
$$L = \frac{1}{T^2}\left|\sum_{k=0}^{1} g_k P_k(f)\right|^2 \left|Q(f)\right|^2 \sum_n \delta(f - n/T) \tag{3.6}$$
$$C = \frac{1}{T}\left[\sum_{k=0}^{1} g_k \left|P_k(f)\right|^2 - \left|\sum_{k=0}^{1} g_k P_k(f)\right|^2 \left|Q(f)\right|^2\right] \quad .$$

The power spectral density has both discrete L and continuous C components that depend on the pulse spectrum and the Fourier transform of the density function used to randomize the signal. Note that when $Q(f)$ is small at multiples of the PRR, the discrete components are small and the spectrum is predominantly continuous. When $Q(f)$ approaches one (negligible dithering) and the bits do not change (e.g., $g_0 = 1$), the continuous spectrum disappears, and the line spectrum dominates. The quantity $g_0 P_0(f) + g_1 P_1(f)$ is the expected value of the pulses.

If bit values are equiprobable (i.e., $g_k = 1/2$) and the pulse representing a 1 is a time delayed version of the pulse representing a 0 (i.e., $p_1(t+\xi) = p_0(t) \equiv p(t)$), equation 3.6 reduces to

$$\bar{R}_{xx}(f) = L + C$$

$$L = \frac{1}{2T^2} |P(f)Q(f)|^2 [1 + \cos(2\pi\xi f)] \sum_n \delta(f - n/T) \tag{3.7}$$

$$C = \frac{1}{T} |P(f)|^2 \left(1 - \frac{|Q(f)|^2 [1 + \cos(2\pi\xi f)]}{2} \right) .$$

When the information bit time delay ξ is small relative to the to the dithering delay (i.e., $\cos(2\pi\xi f) \approx 1$ over the range of frequencies for which $Q(f)$ is significant), the effects of pulse position modulation on the power spectrum are inconsequential.

The results of an example calculation using Equation 3.7 are shown in the following figures. For this example, the signal consists of a short-duration pulse (Figure 3.1) transmitted at a 10 MHz PRR. The dithered pulse position is random and uniformly distributed over 50% of the pulse period. In this calculation, it is assumed that the effects of information bit modulation are negligible over the frequency range of interest. The power spectral density over a frequency range of 1-5000 MHz is shown in Figure 3.2. The magnitude of the spectrum is normalized to the peak of the continuous distribution (at about 250 MHz). The Fourier transform of the density function for this example is $Q(f) = sinc(\pi f T/2)$. This function has nulls at frequencies equal to $2k/T$ ($k = \pm 1, \pm 2, \pm 3, \ldots$), hence the interval between discrete spectral lines is 20 MHz as shown in the figures. For frequencies above 20 MHz, the continuous spectrum is approximately the same as the pulse spectrum (i.e., $P(f)$). Figure 3.3 shows the discrete spectrum over a more limited range (800-1600 MHz) to highlight the individual spectral lines.

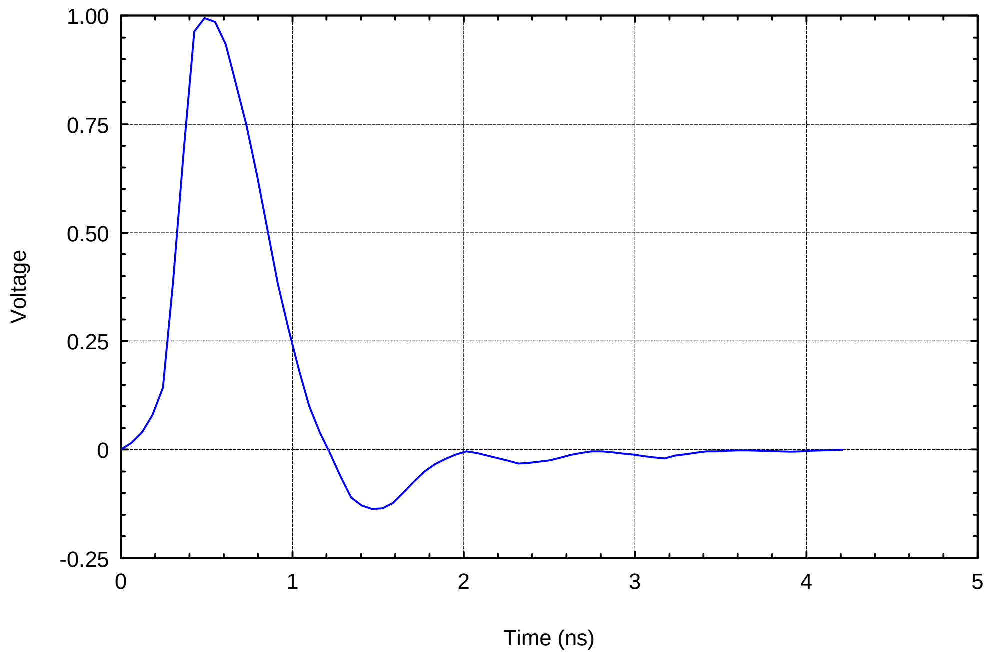

Figure 3.1. Time domain pulse shape.

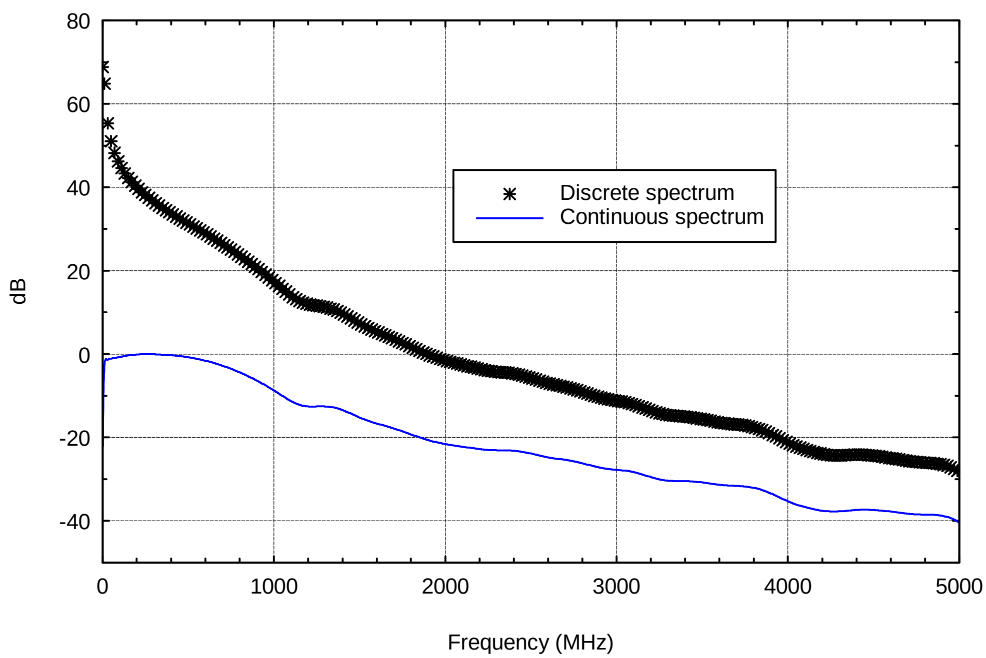

Figure 3.2. Power spectral density for a fixed time-base dithered 10 MHz UWB signal. The pulse positions are uniformly distributed over 50% of the pulse repetition period.

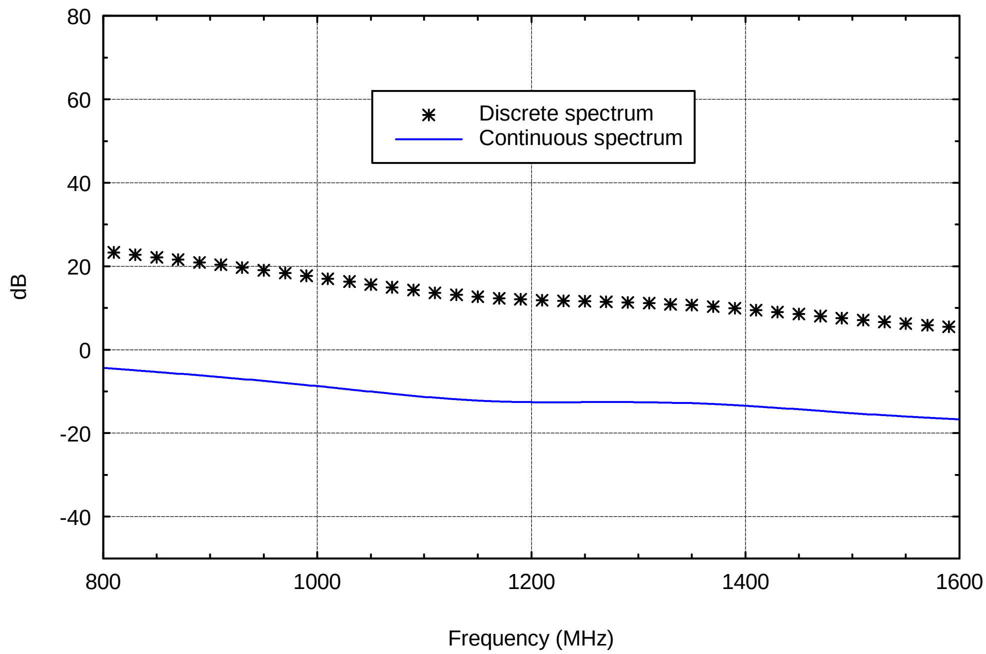

Figure 3.3. Power spectral density showing discrete and continuous spectrum from 800 to 1600 MHz.

The mean power in the bandwidth of a *narrowband* victim RF receiver as a function of frequency can easily be calculated from these results. For example, Figure 3.4 shows the power available to a receiver with a nominal 10 kHz bandwidth. As shown in the figure, the discrete spectrum is not a factor for RF frequencies above a few hundred MHz. For narrowband victim receivers where gains due to the UWB transmitter filters/antenna, propagation channel, and receiver are fairly constant over the receiver bandwidth, the received interference power can easily be calculated by applying the appropriate gain factors to the power in the receiver bandwidth at the center frequency of the receiver.

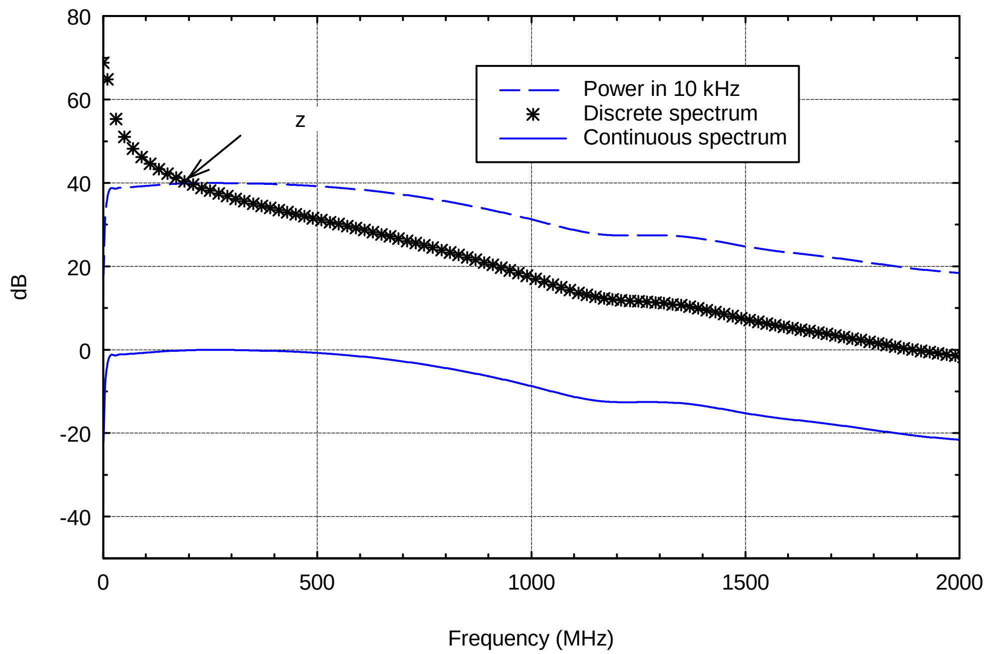

Figure 3.4 Power spectral density showing the continuous spectrum in a 10 kHz bandwidth compared to the discrete spectrum.

3.2.2 Power Spectrum for Finite Duration and Repeated Signals

The results based on Equation 3.1 assume that the signal is on continuously. Obviously, real signals are of finite duration. Also, for some proposed systems, the signal is transmitted for a length of time, say T', and then repeated. In this section, we extend the results presented above to finite duration and repeated signals.

To obtain the power spectrum for a finite duration signal, the following window function

$$w(t) = \begin{cases} 1 & -T' \le t \le T' \\ 0 & else \end{cases}$$
$$W(f) = 2T' sinc(2\pi T' f) \tag{3.8}$$

is multiplied by $x(t)$ (Equation 3.1). The result is that given in Equation 3.7 convolved with the spectrum of the window, i.e., $|W(f)|^2 \circledast \bar{R}_{xx}(f)$ as may be expected. As the window duration increases, the spectrum shape approaches $\bar{R}_{xx}(f)$.

When the series $x(t)$ is windowed and repeated, the autocorrelation function is obtained by taking the expectation of periodic extension of a windowed portion of the series or

$$\mathscr{E}\left\{\sum_{n=-\infty}^{\infty} w(t-nT')x(t-nT') \sum_{m=-\infty}^{\infty} w(s-mT')x(s-mT')\right\} \tag{3.9}$$

The resulting spectrum is

$$\frac{1}{T'^2}\sum_k \bar{R}_{xx}\left(\frac{k}{T'}\right)\circledast\left|W\left(\frac{k}{T'}\right)\right|^2 e^{i2\pi k\tau/T'} \quad , \tag{3.10}$$

which is now discrete with *spectral lines* at frequency intervals of $1/T'$.

3.3 Band Limited Signal Statistics for Fixed Time-base Dithered Systems

From the standpoint of a victim receiver, a fixed time-base dithered UWB signal is a random process. A knowledge of the statistics of such a process is important in predicting how interference affects the performance of a victim receiver. When the UWB PRR is larger than the receiver bandwidth, it may be expected that the received signal would appear to be indistinguishable from Gaussian noise. Since receiver performance in a Gaussian noise environment is well understood, quantifying conditions for which the received UWB interference resembles Gaussian noise is important in predicting receiver performance and developing emissions requirements. Also, when the received signal is Gaussian, only one parameter (mean power) is required to characterize the process. In this section we present the results of an analysis of the fixed time-base dither scheme that can be used to predict when the received UWB signal is approximately Gaussian.

For this analysis, we seek to determine the probability density function that describes the statistics of the UWB signal as seen by the victim receiver (e.g., the final IF stage of the receiver). The following relationship between the density function $a(y)$, its characteristic function $\phi(u)$, and the pulse randomization density function $q(\theta)$ is used to obtain an approximate expression for the received signal statistics

$$\phi(u) = \int e^{iuy} a(y)\,dy = \mathscr{E}\{e^{iux}\} = \int e^{iux(\theta)} q(\theta)\,d\theta \quad . \tag{3.11}$$

Formally, the desired density function is obtained by inserting the UWB signal $x(t)$ (Equation 3.1) into Equation 3.11 and taking the inverse Fourier transform of the characteristic function.

The characteristic function is periodic since the process is cyclostationary as discussed in Section 3.2.1. For purposes of this analysis, the time averaged statistics are obtained by averaging over a period as with the power spectral density function

$$\bar{\phi}(u) = \int_0^T \prod_n \int e^{iup(t-nT-\theta)} q(\theta)\, d\theta \frac{dt}{T} \quad . \tag{3.12}$$

After some manipulations, the density function can be expanded into the well known Edgeworth [1] series. The first four terms of the series are

$$f(x) = \varphi^{(0)}(x) - \frac{\gamma_1}{3!}\varphi^{(3)}(x) + \frac{\gamma_2}{4!}\varphi^{(4)}(x) + \frac{10\gamma_1^2}{6!}\varphi^{(6)}(x) \quad , \tag{3.13}$$

where

$$\varphi^{(n)}(x) = \frac{d^n}{dx^n}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad . \tag{3.14}$$

The desired density function $a(y)$ is related to $f(x)$ by using the transformation $x = (y-m)/\sigma$ where m is the mean and σ is the standard deviation, hence $a(y) = f((y-m)/\sigma)/\sigma$. The first term in the series is the standard normal distribution. The following terms are scaled by coefficients known as the skewness γ_1 and excess γ_2 [1].

In general, the skewness and excess are rather complicated functionals of the pulse shape p and the pulse randomization statistics q. In the case of a narrowband receiver with a center frequency larger than twice the PRR, the expressions are greatly simplified. The following results assume that the power in the spectral lines (if present) is much smaller than that due to the power in the receiver bandwidth due to the continuous spectrum. In addition, if the UWB pulse $P(f)$ spectrum is approximately constant over the bandwidth of the receiver, the variance σ^2, skewness, and excess can be expressed in terms of the baseband impulse response of the receiver filter, $h(t)$, as follows:

$$
\begin{aligned}
&m \approx 0 \\
&\sigma^2 \approx \frac{1}{2T}\int_{-\infty}^{\infty} h^2(t)\,dt = \frac{1}{2T}\int_{-\infty}^{\infty} |H(f)|^2\,df \\
&\gamma_1 \approx 0 \\
&\gamma_2 \approx \frac{3}{4\sigma^4 T}\int_{-\infty}^{\infty}\left[\frac{h^4(t)}{2} - (h^2 \circledast q(t))^2\right]dt
\end{aligned}
\qquad (3.15)
$$

These results show that the variance is proportional to the receiver bandwidth as expected. The mean and skewness are negligible due to the oscillatory characteristics of the bandpass filtered signal. The behavior of the excess as a function of receiver bandwidth was calculated for a receiver with a raised cosine lowpass characteristic and a UWB signal with a 10 MHz PRR. The signal is dithered uniformly over 50% of the pulse repetition period.

Figure 3.5 shows the excess as a funtion of receiver bandwidth. Note that the distribution is approximately Gaussian up to about a 1MHz bandwidth. The excess then decreases to a minimum at about 20 MHz, after which it increases. The normalized distribution for bandwidths below 1 MHz and at10 and 20 MHz are shown in Figure 3.6.

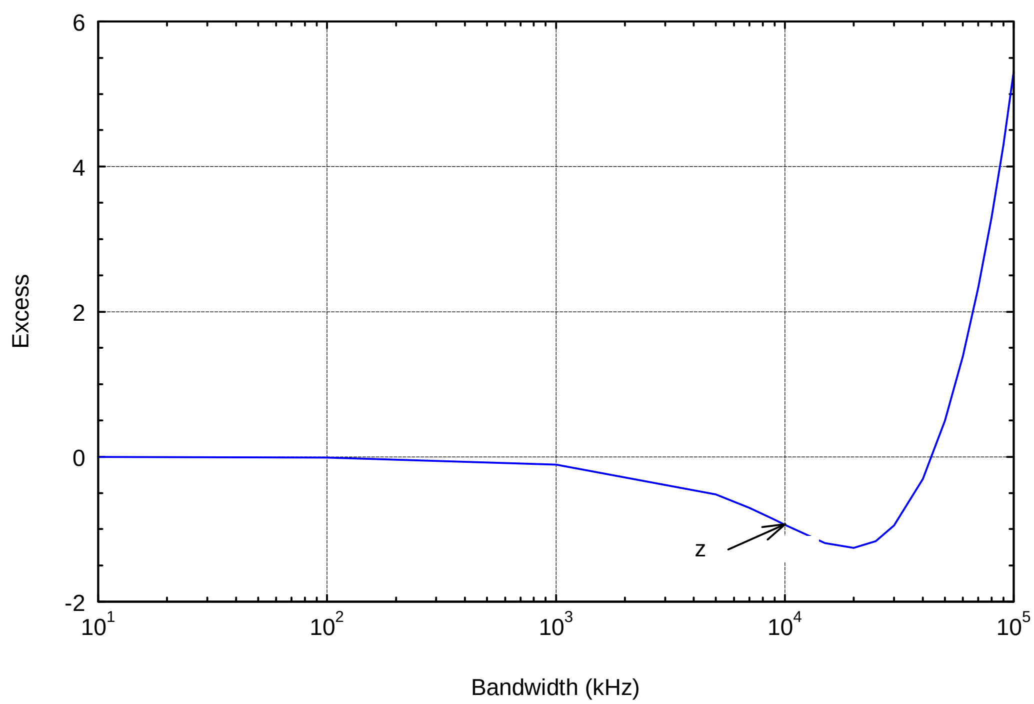

Figure 3.5. The excess as a function of receiver bandwidth.

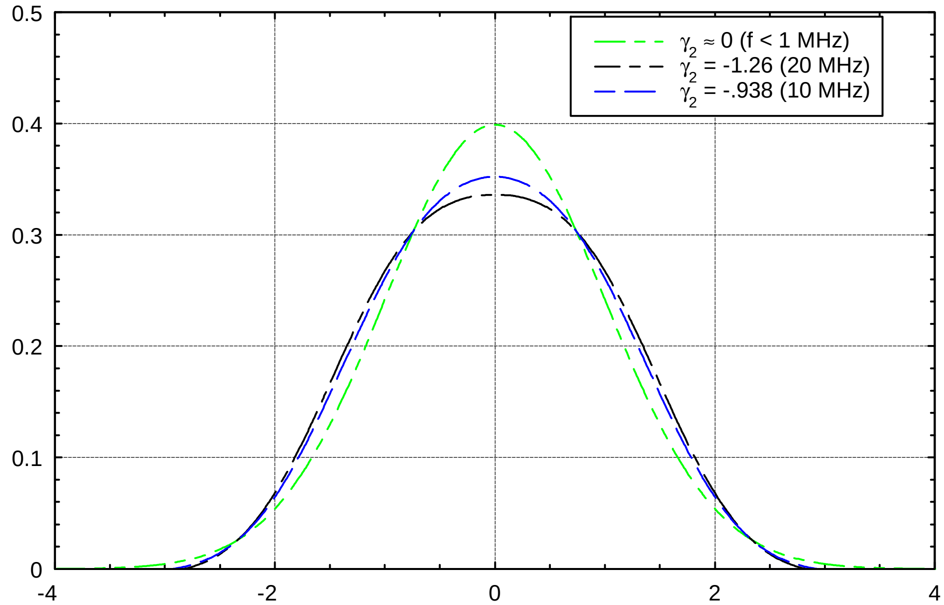

Figure 3.6. The distributions for various receiver bandwidths of less than 1 MHz, and bandwidths of 10 and 20 MHz.

The results presented in this section can be used to predict when an interfering fixed time-base dithered UWB signal is approximately Gaussian in nature, and hence, should be useful in providing guidance to system designers and regulators. Furthermore, as shown in the previous example, they can be used to estimate statistics for bandwidths comparable and exceeding the UWB PRR. In cases where the bandwidth is much larger than the PRR, so that the receiver actually resolves the individual pulses, the results presented above are no longer valid. In such cases, amplitude statistics can readily be estimated by calculating the fraction of time that a particular pulse (as seen by the receiver) amplitude is exceeded during the pulse repetition period.

3.4 References

[1] Harald Cramer, *Mathematical Methods of Statistics*, Princeton NJ: Princeton University Press, 1945.

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